An Introduction to $\mathbb{A}^1$-enumerative geometry

What is enumerative geometry?
= counts solutions to geometric questions

Example: How many lines are there on a smooth cubic surface $X = \{f=0\} \subseteq \mathbb{P}^3$
homogeneous of degree 3

Cayley-Salmon 1849: On every smooth cubic surface there are 27 lines (over $\mathbb{C}$).

Schläfli 1863: There can be 27, 15, 7 or 3 real lines

Segre 1942: there are 2 types of real lines called hyperbolic and elliptic.

$$\#\text{hyperbolic} - \#\text{elliptic} = 3$$

always

Goal for today:
Get something for an arbitrary base field k.

How to count lines on a cubic surface:

$[\ell] \in$
$Gr(2,4) = G(1,3) =$ 2-planes in k^4 / lines in $\mathbb{P}^3$

Let $\pi : V \to Gr(2,4)$ be the vector bundle with fiber

$$V_{[\ell]} = \text{degree 3 polynomials on } \ell$$

$(V = \mathrm{Sym}^3 S^\vee)$

tautological bundle

Let $X = \{f=0\} \subseteq \mathbb{P}^3$ be a cubic surface

degree 3 homogeneous polynomial

f defines a section $\sigma_f : Gr(2,4) \to V$

$$\sigma_f([\ell]) = f|_\ell$$

zeros of $\sigma_f \longleftrightarrow$ lines on X

$$\sum_{\substack{x \in Gr(2,4)(k) \\ \sigma_f(x)=0}} \underset{\circ}{?}(x) = \begin{cases} \sum 1 = 27 & k = \mathbb{C} \\ \sum \pm 1 = 3 & k = \mathbb{R} \end{cases}$$

"local Brouwer degree"

Recall: $f: S^n \to S^n$

$\rightsquigarrow$ $f_*: H_n(S^n) \to H_n(S^n)$

$$\begin{array}{ccc} H_n(S^n) & \to & H_n(S^n) \\ \wr\| & & \wr\| \\ \mathbb{Z} & & \mathbb{Z} \\ 1 & \longmapsto & \deg f \end{array}$$

$\deg f$ = Brouwer degree of f

$$\deg: \underbrace{[S^n, S^n]}_{\text{htpy classes}} \xrightarrow[\simeq]{} \mathbb{Z}$$

$$\underbrace{[f] \longmapsto \deg f}_{\text{Brouwer degree}}$$

local Brouwer degree at a zero x of σ_f:

$k = \mathbb{R}$:

$$\begin{array}{ccccccc} \sigma_f: Gr(2,4) & \longrightarrow & V & & & & B' \\ \cup| & & \cup| & & & & \cap| \\ B & \longrightarrow & V|_B & \cong & B \times \mathbb{R}^4 & \xrightarrow{pr_2} & \mathbb{R}^4 \\ \cup| & & & & & & \\ x & & & & & & \end{array}$$

small ball around x $\to$ B

x $\leftarrow$ zero of σ_f

$$\deg\left(S^4 \cong B/\partial B \longrightarrow B'/\partial B' \cong S^4 \right)$$

← small ball around zero in $\mathbb{R}^d$

$$=: \text{local Brouwer degree}$$

⚠ One needs to be careful with orientations

Brouwer degree in $\mathbb{A}^1$-homotopy theory

Morel's $\mathbb{A}^1$-degree:

$$\deg^{\mathbb{A}^1} : \left[\mathbb{P}^n_k / \mathbb{P}^{n-1}_k , \mathbb{P}^n_k / \mathbb{P}^{n-1}_k \right]_{\mathbb{A}^1} \to GW(k)$$

What is

1) $\mathbb{P}^n_k / \mathbb{P}^{n-1}_k$? ✓

2) $[-,-]_{\mathbb{A}^1}$? ✓

3) $GW(k)$?

Very short introduction to $\mathbb{A}^1$-homotopy theory (=motivic homotopy theory)

Slogan: homotopy theory on algebraic varieties where $\mathbb{A}^1$ plays the role of $[0,1]$

k arbitrary field

Sm_k = category of smooth varieties$/k$

$$\begin{array}{l} Sm_k \\ \big\downarrow \text{Yoneda} \\ sPre(Sm_k) := \text{functors } Sm_k^{op} \to sSet \end{array}$$

$\mathbb{P}^n_k / \mathbb{P}^{n-1}_k$ = pushout $\left(\begin{array}{ccc} \mathbb{P}^{n-1}_k & \to & \mathbb{P}^n_k \\ \downarrow & & \\ * & & \end{array} \right)$

This is a "model category" (or ∞-cat)

In particular, there is a class $\mathcal{W}$ of weak equivalences.

With "Bousfield localization" we can add weak equivalences

- add $X \times \mathbb{A}^1 \to X$
- give varieties a nice topology up to weak eq

$$\mathcal{H}(k) = s\,\mathrm{Pre}(\mathrm{Sm}_k)\,[W^{-1}]$$
$$= \mathbb{A}^1\text{-homotopy category}$$

$$[-,-]_{\mathbb{A}^1} = \mathrm{Hom}_{\mathcal{H}(k)}(-,-)$$

The Grothendieck-Witt ring of k

$GW(k)$ = group completion of isometry classes of finite dimensional non-degenerate symmetric bilinear forms $/k$

$$\beta : V \times V \to k$$

$\oplus$ direct sum = Addition

$\otimes$ makes $GW(k)$ into a ring

generators of $GW(k)$: 1-dimensional forms

$\langle a \rangle : k \times k \to k$ $\quad a \in k^\times / (k^\times)^2$

$(x, y) \mapsto axy$

relations:

1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

2) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$

3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$

$=: h$ hyperbolic form

Examples: • $GW(\mathbb{C}) \cong \mathbb{Z}$

rank

rank, signature $= \# 1\text{'s} - \# -1\text{'s}$ on diag

• $GW(\mathbb{R}) \cong \{ (r,s) \in \mathbb{Z} \times \mathbb{Z} \mid r + s \equiv 0 \mod 2 \}$

$\cong \mathbb{Z} \times \mathbb{Z}$

• $GW(\mathbb{F}_q) \cong \underbrace{\mathbb{Z}}_{\text{rank}} \times \underbrace{\mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2}_{\text{disc} = \det(\text{matrix})} = \mathbb{Z} \times \mathbb{Z}/2$

$\cdot\ GW(\mathbb{Q}_p) \overset{\sim}{=} \dfrac{GW(\mathbb{F}_p) \oplus GW(\mathbb{F}_p)}{\langle h_1 - h \rangle}$

p odd

Springer's thm

generators $\langle a \rangle$ $\quad a \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$

$1, g, p, pg$

$a = a_n p^n + a_{n+1} p^{n+1} + \dots$

$g \neq 1$

$n \in \mathbb{Z}$

$\mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$ is a square iff

- $a_n^{\neq 0}$ square in $\mathbb{F}_p$
- n even

Local $\mathbb{A}^1$-degree

Let $g : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with an isolated zero x

$$\mathbb{P}^n_k \qquad \mathbb{P}^n_k$$

$$\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \qquad \mathbb{P}^n/\mathbb{P}^{n-1}_k$$

Choose a Zariski nbhd U of x

$$\deg^{\mathbb{A}^1}\left(\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k - \{x\}) \simeq U/(U-\{x\}) \xrightarrow{g} \mathbb{A}^n/(\mathbb{A}^n - \{0\}) \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \right)$$

$=:$ local $\mathbb{A}^1$-degree $\deg^{\mathbb{A}^1}_x g$

Theorem: (Kass-Wickelgren)

- If x is a "simple" k-rational zero

  then $\deg^{\mathbb{A}^1}_x g = \langle \det J_g(x) \rangle \in GW(k)$ ← Jacobian

- If x is a simple zero with

  $k(x)/k$ separable FE

$\deg_x^{\mathbb{A}^1} g = \mathrm{Tr}_{k(x)/k} < \det J g(x) > \in GW(k)$

example

$\mathbb{R}$

where

$\mathrm{Tr}_{L/k} B : V \times V \xrightarrow{B} L \xrightarrow{\mathrm{Tr}_{L/k}} k$

field trace

Rmk: There are formulas for when x is an isolated non-simple zero (EKL-form)

(Kass-Wickelgren, Brazelton-Burklund-McKean-Montoro-Opie)

and for when $k(x)$ is <u>not</u> separable over k (Bézoutian)

(Brazelton-McKean-P.)

<u>Theorem</u> (Kass-Wickelgren 2021?):

Let $X = \{f=0\} \subseteq \mathbb{P}^3$ be a smooth cubic surface. Then

$$\sum \mathrm{Tr}_{k(x)/k} \langle \det \mathcal{J}_{\sigma_f}(x) \rangle$$

⚠ watch orientations

x is a zero of σ_f

$$= 15\langle 1 \rangle + 12 \langle -1 \rangle \in GW(k)$$

rank $\to$ 27

signature $\to$ 3

Can do other counts in the same way
just replace $V \to Gr(2,4)$

- lines on a quintic 3-fold (P., M. Levine)
- lines on degree $(2n-1)$-hypersurface in $\mathbb{P}^{n+1}$ (Levine)
- Bézout (McKean)
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